For radial wall jets, Table 2 shows that the standard k-ϵ model produces a spreading rate about 20% below the recommended band of experimental values.[13] For the radial free jet, it can be seen that the model performs somewhat better, as it yields a spreading rate about 8% below the lowest measured value. It is not entirely clear why the present set of results differ from those shown in Table 1. For the radial wall jet, the differences may arise from the fact that Sharma and Patankar[2] used a different set of values for the model coefficients, and also that their spreading rate was deduced by the present author from a very small published graph, a process that inevitably introduces some error. For the radial free jet, Rubel[1] calculated a growth rate 5% in excess of the present calculations using the same k-ϵ model but with a different numerical technique.[15] On the basis of his results it may be argued that the k-ϵ model adequately represents the growth rate of 0.098 measured by Tanaka and Tanaka.[5] However, it is evident that the model predictions are still someway below the respective values of 0.106 and 0.11 measured by Witze and Dwyer[6] and Heskestad.[16]

The plane free jet represents a reliable bench mark by which the accuracy of the present calculation scheme can be judged. For this case, Table 2 shows that the standard k-ϵ model produces a growth rate of 0.104, which is 4% below the value of 0.108 set by Paullay et al.[15] as the standard for parabolic-type calcuations. Other workers[9,10] using parabolic-type marching schemes have computed growth rates in the range of 0.109–0.114.

Table 2 shows that when the ASM[9] is applied to the radial wall jet, the predicted spreading rate is reduced further, so that the agreement with experiment is now worse than that found with the original k-ϵ model. This result is a consequence of the fact that with the ASM the level of shear stress is reduced by the wall-damping model, leading to a decrease in the rate of spread. In contrast, Table 2 shows that the ASM gives very good agreeement with the data for the plane wall jet. If NSA is introduced for this case, the agreement deteriorates somewhat because the spreading rate is increased by about 10%.

The attention is now focused on the results obtained with the modified model that employs NSA. As was previously demonstrated by HL, Table 2 shows that the modified model increases the growth rate of the plane free jet and gives substantial improvement in the calculated rate of spread for the round jet. Turning now to the radial jets, Table 2 shows that the k-ϵ model with NSA predicts generally satisfactory agreement with the measured spreading rates of both types of radial jet. As expected, the introduction of NSA results in an increase in L, leading to greater k levels and greater eddy viscosities and so to an increase in the jet spreading.

Conclusions

The k-ϵ model was modified so that the production rate of k associated with the lateral divergence of the radial flow has a significant influence on the process of scale augmentation. It was demonstrated that the modified model yielded much-improved predictions of radial jets. Further work should consider determining the normal-stress components from an ASM.

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Quadrature Formula for a Double-Pole Singular Integral

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Introduction

In linearized potential thin-wing and airfoil theory, one encounters integrals of the following two forms:

$$I_1(x) = \int_{-1}^{1} (\xi - x)^{-1} f(\xi)\, d\xi, \qquad -1 < x < +1 \tag{1}$$

$$I_2(x) = \int_{-1}^{1} (\xi - x)^{-2} f(\xi)\, d\xi, \qquad -1 < x < +1 \tag{2}$$

In the conventional (Riemann) sense, these integrals are meaningless because of the pole singularity at $x = \xi$. For the origin and context of these important integrals, one may refer to Mangler,[1] who also provides rules for their appropriate interpretation.

The value of these integrals, when they exist, is called the principal value of the concerned integral, when they are interpreted, respectively, in the following limiting sense:

$$\int_{-1}^{1} (\xi - x)^{-1} f(\xi)\, d\xi = \lim_{\epsilon \to 0} \left[\int_{-1}^{x-\epsilon} (\xi - x)^{-1} f(\xi)\, d\xi \right.$$

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$$+ \int_{x+\epsilon}^{1} (\xi - x)^{-1} f(\xi)\, d\xi \Bigg] \tag{3}$$

$$\int_{-1}^{1} (\xi - x)^{-2} f(\xi)\, d\xi = \lim_{\epsilon \to 0} \Bigg[\int_{-1}^{x-\epsilon} (\xi - x)^{-2} f(\xi)\, d\xi$$

$$+ \int_{x+\epsilon}^{1} (\xi - x)^{-2} f(\xi)\, d\xi - 2f(x)/\epsilon \Bigg] \tag{4a}$$

One can show that in those cases where an indefinite integral can be found, the answer can be found by simply inserting the limits $\xi = -1$ and $\xi = +1$, provided any logarithm of $(\xi - x)$ that appears is interpreted as $ln \mid \xi - x \mid$. This important property, in effect, allows the use of conventional integration techniques in many problems. For example, one may write Eq. (4a) in the alternative form

$$\int_{-1}^{1} (\xi - x)^{-2} f(\xi)\, d\xi = \frac{d}{dx} \int_{-1}^{1} (\xi - x)^{-1} f(\xi)\, d\xi \tag{4b}$$

However, there are many other problems where numerical integration of these integrals must be used.

In this brief Note, we show that the generalized quadrature formula derived by Stark[2] for the single-pole singular integral $I_1(x)$ can be extended in a simple way to evaluate the double-pole singular integral $I_2(x)$. The advantage of Stark's formula for $I_1(x)$ is that it is tailormade for integrands containing a weight function $W(x)$ such that $f(x)$ is the product of a regular function and the weight function $W(x)$, where $W(x)$ is assumed to be positive and integrable but not necessarily regular. This advantage is carried over to the extension of Stark's formula for $I_2(x)$.

II. Stark's Formula

Stark[2] provided the following Gaussian quadrature formula for $I_1(x)$ having the weight function $W(x)$:

$$\int_{-1}^{1} (\xi - x)^{-1} f(\xi)\, d\xi = \sum_{i=1}^{N} e_i f(\xi_i)\, (\xi_i - x_j)^{-1} \tag{5}$$

which is exactly valid at the points $x = x_j$ noted subsequently under the conditions that 1) the ratio $f(x)/W(x)$ is a polynomial of degree $\leq 2N$, 2) the points $x = \xi_i$, $i = 1, 2, \ldots, N$ are the N-zeros of the polynomial $P_N(x)$ of degree N in the system $\{P_i(x)\}$ of orthogonal polynomials, assuming $W(x)$ as the weight function in $-1 < x < +1$, 3) the points $x = x_j$, $j = 1, 2, \ldots$, are zeros of the function

$$Q_N(x) = -\frac{1}{2} \int_{-1}^{1} W(\xi) P_N(\xi)(\xi - x)^{-1} d\xi \tag{6}$$

and 4) the coefficients e_i are defined by

$$e_i = [W(\xi_i) P_N'(\xi_i)]^{-1} \int_{-1}^{1} W(\xi) P_N(\xi)(\xi - \xi_i)^{-1} d\xi \tag{7}$$

where $P_N'(\xi_i)$ is the derivative of $P_N(\xi)$ at $x = \xi_i$.

The weights e_i and the abscissas ξ_i are seen to be identical with those of the ordinary Gaussian quadrature formula for the weight function $W(x)$. Closed-form expressions for e_i, ξ_i, x_j for weight functions frequently used in thin-wing and airfoil theory (see Sec. IV) may be found in Ref. 3.

III. Extension of Stark's Formula to $I_2(x)$

For $I_2(x)$, we seek a quadrature formula for the weight function $W(x)$ in the form

$$\int_{-1}^{1} (\xi - x)^{-2} f(\xi) d\xi$$

$$= \sum_{i=1}^{N} e_i [1 - Q_N(x_j)/Q_N(\xi_i)] f(\xi_i)(\xi_i - x_j)^{-2} \tag{8}$$

This formula is exactly valid when $f(x)/W(x)$ is a polynomial of degree $\leq N$ at the points $x = x_j$, $j = 1, 2, \ldots$, provided x_j are now the zeros of the function $Q_N'(x)$, i.e.,

$$Q_N'(x) = -\frac{1}{2} \int_{-1}^{1} W(\xi) P_N(\xi)(\xi - x)^{-2} d\xi$$

$$= -\frac{1}{2} \frac{d}{dx} \int_{-1}^{1} W(\xi) P_N(\xi)(\xi - x)^{-1} d\xi = \frac{d}{dx} Q_N(x) \tag{9}$$

where ξ_i, e_i remain as before for the given weight function $W(x)$.

The proof is quite straightforward. Let

$$f_1(\xi) = \sum_{i=1}^{N} f(\xi_i) h_i(\xi) + A W(\xi) P_N(\xi) \tag{10}$$

where $f_1(\xi)$ is a test function such that $f_1(\xi)/W(\xi)$ is an arbitrary polynomial of degree N; A is a constant, and $h_i(\xi)$ are some interpolating functions defined by

$$h_i(\xi) = P_N(\xi) W(\xi)/[P_N'(\xi_i) W(\xi_i)(\xi - \xi_i)] \tag{11}$$

Equation (8) is valid if it is valid for $f_1(\xi)$, since $f_1(\xi)/W(\xi)$ is an arbitrary polynomial of degree N. Therefore, we substitute $f_1(\xi)$ for $f(\xi)$ on the left-hand side of Eq. (8) and obtain

$$L = \int_{-1}^{1} (\xi - x_j)^{-2} \sum_{i=1}^{N} f(\xi_i) h_i(\xi) d\xi$$

$$+ A \int_{-1}^{1} (\xi - x_j)^{-2} W(\xi) P_N(\xi) d\xi \tag{12}$$

The second integral is zero since it represents $Q_N'(x_j)$.

If we note that

$$(\xi_i - x_j)^2 (\xi - \xi_i)^{-1} (\xi - x_j)^{-2}$$

$$= (\xi - \xi_i)^{-1} - (\xi - x_j)^{-1} - (\xi_i - x_j)(\xi - x_j)^{-2} \tag{13}$$

we may, once again, using $Q_N'(x_j) = 0$, write

$$L = \int_{-1}^{1} (\xi - x_j)^{-2} f_1(\xi) d\xi$$

$$= \sum_{i=1}^{N} e_i [1 - Q_N(x_j)/Q_N(\xi_i)] f(\xi_i)(\xi_i - x_j)^{-2} \tag{14}$$

which completes the proof.

IV. Quadrature Formulas for $I_2(x)$ for Special Weight Functions

In thin-wing and airfoil theory, we frequently encounter the following weight functions: 1) $W(x) = (1 - x^2)^{-1/2}$, 2) $W(x) = (1 - x^2)^{1/2}$, 3) $W(x) = (1 - x)^{1/2} (1 + x)^{-1/2}$, 4) $W(x) = (1 + x)^{1/2} (1 - x)^{-1/2}$. From Ref. 3, their respective e_i, ξ_i, x_j for $I_1(x)$ are as shown in Table 1. The respective $P_N(x)$, $Q_N(x)$, which are all polynomials, are given in terms of the variable $\theta = \cos^{-1} x$, because their roots ξ_i and x_j can be obtained in closed form by simple inspection. The $P_N(x)$ and $Q_N(x)$ are related to Tchebycheff polynomials of the first and second kinds.

With the exception of $W(x) = (1 - x^2)^{1/2}$, for which

$$Q_N'(x) = (N+1)\frac{\pi}{2} \sin (N+1)\theta / \sin\theta$$

Table 1 Summary of quadrature formulas for $I_1(x)$

$W(x)$	$(1-x^2)^{-1/2}$	$(1-x^2)^{1/2}$	$\sqrt{(1-x)/(1+x)}$	$\sqrt{(1+x)/(1-x)}$
$P_N(x)$	$\cos N\theta$	$\sin(N+1)\theta/\sin\theta$	$C\sin(2N+1)(\theta/2)/\sin(\theta/2)$	$C\cos(2N+1)(\theta/2)/\cos(\theta/2)$
$Q_N(x)$	$-\frac{\pi}{2}\sin N\theta/\sin\theta$	$\frac{\pi}{2}\cos(N+1)\theta$	$-\frac{\pi}{2}C\cos(2N+1)(\theta/2)/\cos\theta$	$-\frac{\pi}{2}C\sin(2N+1)(\theta/2)/\sin(\theta/2)$
ξ_i	$\cos[(2i-1)\pi/2N]$, $i=1,2,\ldots,N$	$\cos[(i\pi/(N+1)]$, $i=1,2,\ldots,N$	$\cos[(2i\pi/(2N+1)]$, $i=1,2,\ldots,N$	$\cos[(2i-1)\pi/(2N+1)]$, $i=1,2,\ldots,N$
e_i	$\frac{\pi}{N}(1-\xi_i^2)^{1/2}$	$\frac{\pi}{N+1}(1-\xi_i^2)^{1/2}$	$\frac{2\pi}{2N+1}(1-\xi_i^2)^{1/2}$	$\frac{2\pi}{2N+1}(1-\xi_i^2)^{1/2}$
x_j	$\cos(j\pi/N)$, $j=1,2,\ldots,N-1$	$\cos[(2j-1)\pi/2(N+1)]$, $j=1,2,\ldots,N+1$	$\cos[(2j-1)\pi/(2N+1)]$, $j=1,2,\ldots,N$	$\cos[(2j\pi/(2N+1)]$, $j=1,2,\ldots,N$

$\theta=\cos^{-1}x$; $C=\Gamma(N+1/2)/N!\sqrt{\pi}$; $\Gamma(m)$ is the gamma function with argument m.

Table 2 Comparison of computed and exact solutions of

$$I_2(x)=\int_{-1}^{1}\frac{\xi^4 d\xi}{\sqrt{1-\xi^2}\,(\xi-x)^2},\ -1<x+1$$

x_j	Exact $I_2(x_j)$	Computed $I_2(x_j)$
	For $N=6$	
0.72741239	6.55771764	6.55771764
0.26621648	2.23874179	2.23874179
−0.26621648	2.23874179	2.23874179
−0.72741239	6.55771764	6.55771764

whose roots are

$$x_j=\cos[j\pi/(N+1)],\ j=1,2,\ldots,N$$

the roots of $Q_N'(x)$, unfortunately, do not seem to be extractable in closed form for an arbitrary value of N. Therefore, they must be determined numerically, say, by using the bisection method. We may note, however, that since $Q_N(x)$ is a polynomial, all of whose roots lie in the interval $-1<x<+1$, $Q_N'(x)$, which will be a polynomial of one degree less, will also have all of its roots in the interval $-1<x<+1$. Indeed, between two consecutive roots of $Q_N(x)$ will lie a root of $Q_N'(x)$.

V. Numerical Example of $I_2(x)$

For illustration, we choose to evaluate numerically the integral

$$I_2(x)=\int_{-1}^{1}\frac{\xi^4 d\xi}{\sqrt{1-\xi^2}(\xi-x)^2}=\pi(3x^2+0.5) \qquad (15)$$

whose exact solution is also noted above. The $Q_N'(x)$ for the weight function $(1-x^2)^{-1/2}$ is

$$Q_N'(x)=-\frac{\pi}{4}[(N+1)\sin(N-1)\theta-(N-1)$$

$$\sin(N+1)\theta]/\sin^3\theta$$

where $\theta=\cos^{-1}x$. Let us choose $N=6$, for which we have

$$Q_6'(x)=-\pi(80x^4-48x^2+3)$$

The computed and exact values of $I_2(x)$ are shown in Table 2 and, as expected, they are identical. In passing, we note that integrands having any of the other three weight function, 2–4 in Sec. IV, can always be reinterpreted to have $(1-x^2)^{1/2}$ as its weight function.

VI. Conclusions

Stark's quadrature formula for Cauchy integrals has been extended in a simple manner to evaluate double-pole singular integrals, which occur in linear lifting surface theory.

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Grid Embedding Technique Using Cartesian Grids for Euler Solutions

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Introduction

NUMERICAL solution of the Euler equations is typically carried out by discretization of the flowfield and then solution of the resulting set of coupled equations for each node. Construction of these grids with the requisite smoothness and point clustering remains one of the most difficult tasks associated with the solution process. This is especially true for complex configurations.

The introduction of finite volume formulation made it possible to satisfy the wall boundary conditions, in the integral sense, irrespective of the shape of the boundary. However, a

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